Exercise 70

If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3+2x^2-1}+\frac{b}{x^3+x-2}=0$$

has at least one solution in the interval $(-1,1)$.

Solution

In order to use the Intermediate Value Theorem, the function in question has to be continuous on the closed interval of interest $[-1,1]$. The one on the left side is not because plugging in $x=1$, for example, makes the denominator x^3+x-2 zero. Find the zeros of the polynomials in the denominators in order to factor them.

$$\begin{cases} x^3+2x^2-1=(x+1)(x^2+x-1)=(x+1)\left(x+\dfrac{1+\sqrt{5}}{2}\right)\left(x+\dfrac{1-\sqrt{5}}{2}\right) \\ x^3+x-2=(x-1)(x^2+x+2) \end{cases}$$

As a result, the equation becomes

$$\frac{a}{(x+1)\left(x+\frac{1+\sqrt{5}}{2}\right)\left(x+\frac{1-\sqrt{5}}{2}\right)}+\frac{b}{(x-1)(x^2+x+2)}=0.$$

Multiply both sides by $(x+1)(x-1)\left(x+\frac{1-\sqrt{5}}{2}\right)$ to remove all factors in the denominators that are zero within the interval $[-1,1]$.

$$\frac{a(x-1)}{x+\frac{1+\sqrt{5}}{2}}+\frac{b(x+1)\left(x+\frac{1-\sqrt{5}}{2}\right)}{x^2+x+2}=0$$

Let $f(x)$ be this function on the left side.

$$f(x)=\frac{a(x-1)}{x+\frac{1+\sqrt{5}}{2}}+\frac{b(x+1)\left(x+\frac{1-\sqrt{5}}{2}\right)}{x^2+x+2}$$

Find a value of x for which the function is negative, and find a value of x for which the function is positive.

$$f(0)=-\frac{2a}{1+\sqrt{5}}-\frac{1}{4}(\sqrt{5}-1)b<0$$

$$f(1)=\frac{1}{4}(3-\sqrt{5})b>0$$

$f(x)$ is the sum of two rational functions, which is continuous on the desired interval $[-1,1]$ since all values of x that made the denominators zero have been removed. $N=0$ lies between $f(0)$ and $f(1)$. By the Intermediate Value Theorem, then, there exists a root within $(0,1)$. Therefore, there is at least one solution in the interval $(-1,1)$.